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ABSTRACT: We explain the connection between the different approaches taken in [Be] and [Gr] in examining the tame kernel $K_2(O_k)$, where O_k is the ring of integers of a number field k. In particular, we generalize the normic criterion in [Be, 2.4] by using Jaulent's work which gives links between K_2 and S-genera.

1. INTRODUCTION

From [Ta] we have a p^n-rank formula for $K_2(O_k)$, if k contains the p^n-th roots of unity. There are many publications that investigate the p^n-rank, in particular for $p = 2$ see for example [Ko]. It follows from [Ta] that the 2-rank of $K_2(O_k)$ is at least r_1, the number of real infinite places of k. Therefore the 2-primary subgroup of $K_2(O_k)$ is smallest if it is elementary abelian of that rank: $2\text{-}primK_2(O_k) \cong (\mathbb{Z}/2)^{r_1}$. The totally real number fields with this property are examined in [Be], where they are denoted as having property (*). The methods used in [Be] are based on the work of Conner and Hurrelbrink, using the exact hexagon defined in [C-H_2] to examine the interplay between units, class numbers and ramification. The hexagon involves galois cohomology groups; parts of it were already introduced in [C-H_1], where one result is: "*Let k be an arbitrary number field. The 2-Sylow subgroup of $K_2(O_k)$ is elementary abelian of rank r_1 iff 2 does not split in $k/\mathbb{Q}$, the S-class number of k is odd, and k contains S-units with independent signs, where S is the set of infinite and dyadic places of k.*"

A characterization of these number fields has been obtained in [Gr], even for arbitrary primes, by examining p-ramification. In [Gr] the tame kernel is defined in a new sense (the narrow one), which is more convenient for the examination of the 2-part. Using the notation suggested in [Ke], we denote it by $K_2^+(O_k)$. It is related to the classical tame kernel by: $K_2(O_k)/K_2^+(O_k) \cong (\mathbb{Z}/2)^{r_1}$. The connection between p-ramification and K_2 is given by [Gr, Thm 1]: "*Let k be a number field and p a prime. If the maximal real subfield of $\mathbb{Q}(\mu_p)$ is contained in k, then $dim_p K_2^+(O_k) = dim_p T + \delta$; where δ is the defect of Leopoldt conjecture at p in k, and T is the finite p-group which, in the maximal abelian p-extension unramified outside p, fixes the composite of all $\mathbb{Z}_p$-extensions of k.*" The results in [Gr] assume the validity of the Leopoldt conjecture, but this restriction is lifted in [G-J]. There, the notion of p-regular fields is introduced and the Leopoldt conjecture is verified for certain galois p-extensions of regular fields.

Definition: A number field k is *p-regular* if the p-primary subgroup of $K_2^+(O_k)$ is trivial.

In [Gr, Cor to Thm 1] a criterion for the triviality of the p-part of $K_2^+(O_k)$ in terms of ideal class groups is derived: "*Let k be a number field that contains the p-th roots of unity. Then k is p-regular iff p does not split in $k/\mathbb{Q}$ and the p-primary subgroup of the narrow S-class group of k is trivial, where S is the set of places of k lying over p.*" More equivalent criteria are given in [G-J, 2.1].

For $p = 2$ one obtains the above criterion from [C-H$_1$], but to see this we need to clarify a few definitions. The ideal class groups used in [Gr] are the ones that are classically referred to as narrow ideal class groups. That is, they are the quotient of the group of fractional ideals of k by the subgroup of principal ideals with totally positive generators. Similarly, the group of units $U_k = \{x \in k^\times \mid v_P(x) = 0 \text{ for all finite places } P\}$ in [Gr] includes the corresponding restriction at infinite places. The resulting group of totally positive units will be referred to as U_k^+. Let S be a set of places of k. The S-ideal class group $Cl_{k,S}$ is the quotient of the ideal class group Cl_k by $Cl_k(S)$, the subgroup generated by classes of places in S. The group of S-units is $U_{k,S} = \{x \in k^\times \mid v_P(x) = 0 \text{ for all places } P \notin S\}$. Again, excluding infinite places from S results in the narrow S-ideal class group $Cl_{k,S}^+$, and the totally positive S-units $U_{k,S}^+$. The following exact sequence relates $Cl_{k,S}^+$ to $Cl_{k,S}$ (note that $k^\times/\Big((k^\times)^+ \cdot U_{k,S}\Big) = 1$ means that k contains S-units with independent signs):

$$1 \to k^\times/\Big((k^\times)^+ \cdot U_{k,S}\Big) \to Cl_k^+/Cl_k^+(S) \to Cl_k/Cl_k(S) \to 1$$

2. NORMIC CRITERION

Since property (*) in [Be] is only a special case of 2-regularity, some of the explicit results in [Be] can be obtained by interpreting the results from [Gr] in the case $p = 2$. Some, however, do not follow in a straightforward manner. For example, the normic condition in the lift theorem [Be, 2.4] can be derived using S-genera theory. It follows as a special case of the following theorem. We will assume μ_p to be contained in the base field, otherwise the notion of p-regular would have to be changed into p-rational; see [G-J], [J-N] and [M-N].

Theorem: *Let $p \geq 2$ be a prime, and let F be a number field that contains μ_p. Let E/F be a Galois p-extension, and let S be the set of places lying over p, in F or E respectively. Then, E is p-regular if and only if: F is p-regular, p does not split in E, and $\Big(U_{F,S}^+ : U_{F,S}^+ \cap N_{E/F}(E^*)\Big) = \prod_{P\in T} e_P^{ab}(E/F)$; where the product is taken over T the set of all tame ramified primes P in E/F, and $e_P^{ab}(E/F)$ indicates the ramification index of P in the maximal abelian sub-extension in E_P/F_P.*

Proof: By the above characterization of p-regularity, we need to show that under the assumtion that p does not split in E we have: $p\text{-}prim\ Cl_{E,S}^+ = 1 \Leftrightarrow p\text{-}prim\ Cl_{F,S}^+ = 1$ and $\Big(U_{F,S}^+ : U_{F,S}^+ \cap N_{E/F}(E^*)\Big) = \prod_{P\in T} e_P^{ab}(E/F)$. If we let ${}_pCl$ denote the p-primary subgroup of Cl, we have from [Ja, Thm. III.2.12]:

$$|{}_pCl_{E,S}^{+G}| = |{}_pCl_{F,S}^+| \cdot \frac{\prod_{P\in S} d_P^{ab}}{[E^{ab} : F]} \cdot \frac{\prod_{P\in T} e_P^{ab}(E/F)}{\Big(U_{F,S}^+ : U_{F,S}^+ \cap N_{E/F}(E^*)\Big)} \cdot |\kappa_{E/F}| \ ,$$

where $d_P^{ab}(E/F)$ is the local degree and $e_P^{ab}(E/F)$ the ramification index in the maximal abelian sub-extension of E_P/F_P, and $\kappa_{E/F}$ is the knot number. If p does not split in E,

$\kappa = 1$ by [G-J, 2.5], and the above simplifies to:

$$|{}_pCl^{+G}_{E,S}| = |{}_pCl^{+}_{F,S}| \cdot 1 \cdot \frac{\prod_{P \in T} e_P^{ab}(E/F)}{\left(U^+_{F,S} : U^+_{F,S} \cap N_{E/F}(E^*)\right)} \cdot 1$$

□

For computational purposes, note that the p-rank of $U^+_{F,S}/(U^+_{F,S})^p$ is $r_2(F)+1$, hence $\left(U^+_{F,S} : U^+_{F,S} \cap N_{E/F}(E^*)\right) \leq r_2(F)+1$. Furthermore, the concept of primitive ramification may yield a more practically computable criterion; for the definitions we refer to [G-J, 1.1] and [G-J, 1.3]. From this point of view we obtain by [G-J, 2.3]:

Corollary: *Let F be a number field that contains μ_p, and let E/F be a Galois p-extension. If F is p-regular then: the set T of tame ramified primes in E/F is p-primitive if and only if p does not split in E/F and* $\left(U^+_{F,S} : U^+_{F,S} \cap N_{E/F}(E^*)\right) = \prod_{P \in T} e_P^{ab}(E/F)$.
□

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