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On the global and weak dimensions of pullbacks of non-commutative rings

N. KOSMATOV

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Nikolai Kosmatov

All rings will be assumed to have identity elements preserved by ring homomorphisms, and all modules, unless specified otherwise, will be left modules. For a ring S, $\operatorname{lgld} S$ and $\operatorname{lwd} S$ will denote the left global dimension of S and the left weak global dimension of S, respectively. For an S-module X and a right S-module Y, the projective dimension of X, the injective dimension of X, the flat dimension of X, and the flat dimension of Y are denoted by $\operatorname{pd}_S X$, $\operatorname{id}_S X$, $\operatorname{fd}_S X$ and $\operatorname{rfd}_S Y$, respectively.

A commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow{j_2} & R' \end{array} \tag{1}$$

is said to be a pullback (or a cartesian square, or a fibre product) if given $r_1 \in R_1$, $r_2 \in R_2$ with $j_1(r_1) = j_2(r_2)$ there is a unique element $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$ (note that if j_2 is a surjection then so is i_1, but not conversely). The ring R is called the fibre product (or pullback) of R_1 and R_2 over R'.

Assuming that j_2 is surjective, Milnor [5, Chapter 2] characterized projective modules over such a ring R. Facchini and Vámos [2] established analogues of Milnor's theorems for injective and flat modules. Kirkman and Kuzmanovich [3, Theorem 2] showed that if (1) is a pullback square with j_2 surjective, then

$$\operatorname{lgld} R \leqslant \max_{k=1,2}\{\operatorname{lgld} R_k + \operatorname{rfd}_R R_k\}. \tag{2}$$

For commutative rings, Scrivanti [6] sharpened this upper bound on $\operatorname{lgld} R$ and obtained an upper bound on $\operatorname{lwd} R$. Moreover, she gave examples which showed that, in a certain case, those results were best possible. Besides, Cowley showed that it could be beneficial to work only on one side of the rings in question [1, Example 3.4]. He obtained the following "one-sided" bound [1, Theorem 3.1]: if (1) is a pullback square with j_2 surjective, then

$$\operatorname{lgld} R \leqslant \max_{k=1,2}\{\operatorname{lgld} R_k + \operatorname{pd}_R R_k\}. \tag{3}$$

Our Theorems 9, 10 and Corollaries 12, 13 are the generalizations to non-commutative rings of Scrivanti's results. In Theorem 8, we provide a "one-sided" bound on the left global dimension of a pullback ring. In Propositions 5, 6 and 7, we give sufficient conditions for an R-module M to have injective, projective and flat dimensions $\leqslant n$. In Corollaries 11 and 12, we deduce the upper bounds (3) and (2) as immediate consequences of our Propositions 5 and 6. Here we relax the conditions and only require i_1 to be surjective.

We begin with the following consequence of [2, Theorem 2].

Theorem 1. *Let (1) be a pullback diagram with i_1 surjective. Then an R-module M is injective (projective, flat) if and only if* $\operatorname{Hom}_R(R_1, M)$ *and* $\operatorname{Hom}_R(R_2, M)$ *($R_1 \otimes_R M$ and $R_2 \otimes_R M$) are R_1- and R_2-injective (projective, flat) modules respectively.*

Proof. Set $R'' = j_2(R_2)$. Since i_1 is a surjection, we obtain $j_1(R_1) \subset j_2(R_2) = R''$. Thus we have another commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow{j_2} & R''. \end{array}$$

It is clear that this diagram is a pullback square with $j_2 : R_2 \to R''$ surjective. So the desired result follows from [2, Theorem 2].

Proposition 2. *Let M be an R-module, n be a positive integer, and let*

$$0 \longrightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \longrightarrow \dots$$

be an injective resolution of M. Let K_t denote $\operatorname{im}(f_{t+1})$, $t \geqslant 0$. Suppose that $\operatorname{id}_{R_k}(\operatorname{Ext}^l_R(R_k, M)) \leqslant n - l$ for $l = 0, 1, \dots, n$ and $k = 1, 2$. Then $\operatorname{id}_{R_k}(\operatorname{Hom}_R(R_k, K_t)) \leqslant n - t - 1$ for $t = 0, 1, \dots, n$ and $k = 1, 2$.

Proof. For any $n \geqslant 1$ and $k = 1, 2$, the proof is by induction on t.

For $t = 0$, if we apply the functor $\mathrm{Ext}^*_R(R_k, -)$ to the short exact sequence of R-modules $0 \longrightarrow M \longrightarrow I_0 \xrightarrow{f_1} K_0 \longrightarrow 0$, we obtain an exact sequence of R_k-modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(R_k, M) & \longrightarrow & \mathrm{Hom}_R(R_k, I_0) & \xrightarrow{f_{1*}} & \\ & \xrightarrow{f_{1*}} & \mathrm{Hom}_R(R_k, K_0) & \longrightarrow & \mathrm{Ext}^1_R(R_k, M) & \longrightarrow & 0 \end{array} \tag{4}$$

and isomorphisms of R_k-modules

$$\mathrm{Ext}^l_R(R_k, K_0) \simeq \mathrm{Ext}^{l+1}_R(R_k, M), \ l \geqslant 1. \tag{5}$$

Set $A_{k,0} = \mathrm{im}\, f_{1*}$ and break up (4) into two short exact sequences of R_k-modules:

$$0 \longrightarrow \mathrm{Hom}_R(R_k, M) \longrightarrow \mathrm{Hom}_R(R_k, I_0) \xrightarrow{f_{1*}} A_{k,0} \longrightarrow 0, \tag{6}$$

$$0 \longrightarrow A_{k,0} \hookrightarrow \mathrm{Hom}_R(R_k, K_0) \longrightarrow \mathrm{Ext}^1_R(R_k, M) \longrightarrow 0. \tag{7}$$

Since I_0 is an injective R-module, it can easily be checked that R_k-module $\mathrm{Hom}_R(R_k, I_0)$ is injective. Since $\mathrm{id}_{R_k}(\mathrm{Hom}_R(R_k, M)) \leqslant n$, we obtain from (6) that $\mathrm{id}_{R_k}(A_{k,0}) \leqslant n - 1$. At the same time $\mathrm{id}_{R_k}(\mathrm{Ext}^1_R(R_k, M)) \leqslant n - 1$. Therefore, using (7), we get $\mathrm{id}_{R_k}(\mathrm{Hom}_R(R_k, K_0)) \leqslant n - 1$.

For $t \geqslant 1$, we apply the functor $\mathrm{Ext}^*_R(R_k, -)$ to the short exact sequence of R-modules $0 \longrightarrow K_{t-1} \hookrightarrow I_t \xrightarrow{f_{t+1}} K_t \longrightarrow 0$. We get an exact sequence of R_k-modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(R_k, K_{t-1}) & \longrightarrow & \mathrm{Hom}_R(R_k, I_t) & \xrightarrow{(f_{t+1})_*} & \\ & \xrightarrow{(f_{t+1})_*} & \mathrm{Hom}_R(R_k, K_t) & \longrightarrow & \mathrm{Ext}^1_R(R_k, K_{t-1}) & \longrightarrow & 0 \end{array} \tag{8}$$

and isomorphisms of R_k-modules

$$\mathrm{Ext}^l_R(R_k, K_t) \simeq \mathrm{Ext}^{l+1}_R(R_k, K_{t-1}), \ l \geqslant 1. \tag{9}$$

Put $A_{k,t} = \mathrm{im}\, (f_{t+1})_*$ and break up (8) into two short exact sequences of R_k-modules:

$$0 \longrightarrow \mathrm{Hom}_R(R_k, K_{t-1}) \longrightarrow \mathrm{Hom}_R(R_k, I_t) \xrightarrow{(f_{t+1})_*} A_{k,0} \longrightarrow 0, \tag{10}$$

$$0 \longrightarrow A_{k,t} \hookrightarrow \mathrm{Hom}_R(R_k, K_t) \longrightarrow \mathrm{Ext}^1_R(R_k, K_{t-1}) \longrightarrow 0. \tag{11}$$

By the inductive hypothesis, we have $\mathrm{id}_{R_k}(\mathrm{Hom}_R(R_k, K_{t-1})) \leqslant n - t$. Since I_t is an injective R-module, $\mathrm{Hom}_R(R_k, I_t)$ is an injective R_k-module. Therefore, from (10), $\mathrm{id}_{R_k}(A_{k,t}) \leqslant n - t - 1$. Combining (5) and (9), we have $\mathrm{Ext}^1_R(R_k, K_{t-1}) \simeq \ldots \simeq \mathrm{Ext}^t_R(R_k, K_0) \simeq \mathrm{Ext}^{t+1}_R(R_k, M)$. Hence $\mathrm{id}_{R_k}(\mathrm{Ext}^1_R(R_k, K_{t-1})) = \mathrm{id}_{R_k}(\mathrm{Ext}^{t+1}_R(R_k, M)) \leqslant n-t-1$. Finally, from (11), we obtain $\mathrm{id}_{R_k}(\mathrm{Hom}_R(R_k, K_t)) \leqslant n - t - 1$, as required.

The proofs of the following Propositions 3 and 4 are similar to that of Proposition 2 if we apply the functors $\mathrm{Tor}^R_*(R_k, -)$ to the given resolutions.

Proposition 3. *Let M be an R-module, n be a positive integer, and let*

$$\ldots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a projective resolution of M. Let K_t denote $\ker f_t$, $t \geqslant 0$. Suppose that $\mathrm{pd}_{R_k}(\mathrm{Tor}^R_l(R_k, M)) \leqslant n - l$ for $l = 0, 1, \ldots, n$ and $k = 1,2$. Then $\mathrm{pd}_{R_k}(R_k \otimes_R K_t) \leqslant n - t - 1$ for $t = 0, 1, \ldots, n$ and $k = 1,2$.

Proposition 4. *Let M be an R-module, n be a positive integer, and let*

$$\ldots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a flat resolution of M. Let K_t denote $\ker f_t$, $t \geqslant 0$. Suppose that $\mathrm{fd}_{R_k}(\mathrm{Tor}^R_l(R_k, M)) \leqslant n - l$ for $l = 0, 1, \ldots, n$ and $k = 1,2$. Then $\mathrm{fd}_{R_k}(R_k \otimes_R K_t) \leqslant n - t - 1$ for $t = 0, 1, \ldots, n$ and $k = 1,2$.

Proposition 5. *Let (1) be a pullback diagram with i_1 surjective, M be an R-module, and let n be a non-negative integer. Suppose that $\mathrm{id}_{R_k}(\mathrm{Ext}^l_R(R_k, M)) \leqslant n-l$ for $l = 0, 1, \ldots, n$ and $k = 1,2$. Then $\mathrm{id}_R M \leqslant n$.*

Proof. For the case $n = 0$, the result follows from Theorem 1. For $n \geqslant 1$, consider an injective resolution of R-module M

$$0 \longrightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \longrightarrow \ldots .$$

By definition, put $K_t = \mathrm{im}(f_{t+1})$ for $t \geqslant 0$. From Proposition 1, we get that R_k-module $\mathrm{Hom}_R(R_k, K_{n-1})$ is injective ($k = 1,2$). Hence, by Theorem 1, K_{n-1} is an injective R-module. This means that $\mathrm{id}_R M \leqslant n$.

Arguing as above, the reader will easily prove the following analogues of Proposition 5 for projective and flat dimensions.

Proposition 6. *Let (1) be a pullback diagram with i_1 surjective, M be an R-module, and let n be a non-negative integer. Suppose that* $\operatorname{pd}_{R_k}(\operatorname{Tor}_l^R(R_k, M)) \leqslant n - l$ *for* $l = 0, 1, \ldots, n$ *and* $k = 1, 2$. *Then* $\operatorname{pd}_R M \leqslant n$.

Proposition 7. *Let (1) be a pullback diagram with i_1 surjective, M be an R-module, and let n be a non-negative integer. Suppose that* $\operatorname{fd}_{R_k}(\operatorname{Tor}_l^R(R_k, M)) \leqslant n - l$ *for* $l = 0, 1, \ldots, n$ *and* $k = 1, 2$. *Then* $\operatorname{fd}_R M \leqslant n$.

Proposition 5 clearly implies the following theorem.

Theorem 8. *Let (1) be a pullback diagram with i_1 surjective, and let n be a non-negative integer. Suppose that for any R-module M we have that*

$$\operatorname{id}_{R_k}(\operatorname{Ext}_R^l(R_k, M)) \leqslant n - l \quad \textit{for} \quad l = 0, 1, \ldots, n \quad \textit{and} \quad k = 1, 2.$$

Then $\operatorname{lgld} R \leqslant n$.

Theorem 9. *Let (1) be a pullback diagram with i_1 surjective, and let n be a non-negative integer. Suppose that for any left ideal J of R we have that*

$$\operatorname{pd}_{R_k}(\operatorname{Tor}_l^R(R_k, R/J)) \leqslant n - l \quad \textit{for} \quad l = 0, 1, \ldots, n \quad \textit{and} \quad k = 1, 2.$$

Then $\operatorname{lgld} R \leqslant n$.

Proof. Let J be a left ideal of R. Proposition 6 shows immediately that $\operatorname{pd}_R(R/J) \leqslant n$. Therefore, using Auslander's theorem, we have $\operatorname{lgld} R = \sup\{\operatorname{pd}_R(R/J) \mid J \text{ is a left ideal of } R\} \leqslant n$.

Theorem 10. *Let (1) be a pullback diagram with i_1 surjective, and let n be a non-negative integer. Suppose that for any finitely generated left ideal J of R we have that*

$$\operatorname{fd}_{R_k}(\operatorname{Tor}_l^R(R_k, R/J)) \leqslant n - l \quad \textit{for} \quad l = 0, 1, \ldots, n \quad \textit{and} \quad k = 1, 2.$$

Then $\operatorname{lwd} R \leqslant n$.

Proof. Let J be a finitely generated left ideal of R. Proposition 7 evidently implies that $\operatorname{fd}_R(R/J) \leqslant n$. Therefore, since $\operatorname{lwd} R = \sup\{\operatorname{pd}_R(R/J) \mid J \text{ is a finitely generated left ideal of } R\}$, we have $\operatorname{lgld} R \leqslant n$.

Corollary 11. *If (1) is a pullback diagram with i_1 surjective, then*

$$\operatorname{lgld} R \leqslant \max_{k=1,2}\{\operatorname{lgld} R_k + \operatorname{pd}_R R_k\}.$$

Proof. Set $n_k = \operatorname{lgld} R_k$, $m_k = \operatorname{pd}_R R_k$, $N_k = n_k + m_k$ $(k = 1, 2)$ and $N = \max\{N_1, N_2\}$. It can be assumed that $m_k, n_k < \infty$. Let M be an R-module and $k \in \{1, 2\}$. Since $\operatorname{pd}_R R_k = m_k$, we have $\operatorname{Ext}^l_R(R_k, M) = 0$ for all $l \geqslant m_k + 1$. At the same time, since $\operatorname{lgld} R_k = n_k$, we get $\operatorname{id}_{R_k}(\operatorname{Ext}^l_R(R_k, M)) \leqslant n_k = N_k - m_k \leqslant N_k - l \leqslant N - l$ for any $l = 0, 1, \ldots, m_k$. Therefore, by Proposition 5, $\operatorname{pd}_R M \leqslant N$. This means that $\operatorname{lgld} R \leqslant N$.

Similarly, Propositions 6 and 7 allow us to prove Corollaries 12 and 13.

Corollary 12. *If (1) is a pullback diagram with i_1 surjective, then*

$$\operatorname{lgld} R \leqslant \max_{k=1,2}\{\operatorname{lgld} R_k + \operatorname{rfd}_R R_k\}.$$

Corollary 13. *If (1) is a pullback diagram with i_1 surjective, then*

$$\operatorname{lwd} R \leqslant \max_{k=1,2}\{\operatorname{lwd} R_k + \operatorname{rfd}_R R_k\}.$$

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Equipe de Mathématiques
U.M.R. 6623 du C.N.R.S.
16, route de Gray
25030 Besançon Cedex France

Department of Mathematics and Mechanics
Saint-Petersburg State University
Bibliotechnaya pl. 2
Saint-Petersburg, 198904, Russia

E-Mail: koko@nk1442.spb.edu