THÉORIE DES NOMBRES
BESANÇON

Années 1998/2001

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In this article, all rings are assumed to have identity elements preserved by ring homomorphisms, and all modules are left modules. For a ring Λ, let $\operatorname{lgld}\Lambda$ and $\operatorname{wd}\Lambda$ denote the left global dimension of Λ and the weak dimension of Λ, respectively. For a Λ-module X, we denote the injective, projective and flat dimensions of X by $\operatorname{id}_\Lambda X$, $\operatorname{pd}_\Lambda X$ and $\operatorname{fd}_\Lambda X$, respectively. The left finitistic injective, projective and flat dimensions of Λ are denoted and defined as follows:

$$\begin{aligned} \operatorname{lFID}\Lambda &= \sup\{\operatorname{id}_\Lambda M \mid M \text{ is a } \Lambda\text{-module with } \operatorname{id}_\Lambda M < \infty\}, \\ \operatorname{lFPD}\Lambda &= \sup\{\operatorname{pd}_\Lambda M \mid M \text{ is a } \Lambda\text{-module with } \operatorname{pd}_\Lambda M < \infty\}, \\ \operatorname{lFFD}\Lambda &= \sup\{\operatorname{fd}_\Lambda M \mid M \text{ is a } \Lambda\text{-module with } \operatorname{fd}_\Lambda M < \infty\}. \end{aligned}$$

Consider a commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow{j_2} & R', \end{array} \tag{1}$$

where R is the pullback (also called fibre product) of R_1 and R_2 over R', that is, given $r_1 \in R_1$ and $r_2 \in R_2$ with $j_1(r_1) = j_2(r_2)$, there is a unique element $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$. We assume that i_1 is a surjection.

In the present paper we continue our study of homological dimensions of pullbacks started in [1]. Our purpose is to give upper bounds for the finitistic dimensions of R (Theorems 1, 2 and 3). We also provide two simple examples of pullbacks where we use these results to calculate homological dimensions, and show that our conditions are essential. In the first example, a pullback of two hereditary rings has finite finitistic dimensions though its global and weak dimensions are infinite. Therefore, it is impossible to estimate the global and weak dimensions of a pullback if only that of the component rings are given. The second example demonstrates that our estimates would not be true if we dropped the assumption that i_1 is surjective.

Theorem 1. *Let n be a non-negative integer. Suppose that for every R-module M of finite injective dimension we have that*

$$\mathrm{id}_{R_k}(\mathrm{Ext}^l_R(R_k, M)) \leqslant n - l \quad \textit{for} \quad l = 0, 1, \ldots, n \quad \textit{and} \quad k = 1, 2.$$

Then $\mathrm{lFID}\, R \leqslant n$.

Proof. Let M be an R-module of finite injective dimension. From [1, Proposition 5] it follows that $\mathrm{id}_R M \leqslant n$. Therefore $\mathrm{lFID}\, R = \sup\{\mathrm{id}_R M \mid \mathrm{id}_R M < \infty\} \leqslant n$.

Similarly, [1, Propositions 6 and 7] allow us to prove analogous bounds for finitistic projective and flat dimensions.

Theorem 2. *Let n be a non-negative integer. Suppose that for every R-module M of finite projective dimension we have that*

$$\mathrm{pd}_{R_k}(\mathrm{Tor}^R_l(R_k, M)) \leqslant n - l \quad \textit{for} \quad l = 0, 1, \ldots, n \quad \textit{and} \quad k = 1, 2.$$

Then $\mathrm{lFPD}\, R \leqslant n$.

Theorem 3. *Let n be a non-negative integer. Suppose that for every R-module M of finite flat dimension we have that*

$$\mathrm{fd}_{R_k}(\mathrm{Tor}^R_l(R_k, M)) \leqslant n - l \quad \textit{for} \quad l = 0, 1, \ldots, n \quad \textit{and} \quad k = 1, 2.$$

Then $\mathrm{lFFD}\, R \leqslant n$.

Example 1. Let $s \geqslant 2$, $R' = \mathbb{Z}/s\mathbb{Z}$, $R_1 = R_2 = \mathbb{Z}$, $R = \{\, (m_1, m_2) \in R_1 \times R_2 \mid m_1 \equiv m_2\, (mod\, s) \,\}$. Then in the commutative square (1) with canonical surjections i_k and j_k the ring R is the pullback of R_1 and R_2 over R'. There exist the periodic free resolutions of the R-modules R_k

$$\ldots \xrightarrow{(0,s)} R \xrightarrow{(s,0)} R \xrightarrow{(0,s)} R \xrightarrow{i_1} R_1 \longrightarrow 0, \tag{2}$$

$$\ldots \xrightarrow{(s,0)} R \xrightarrow{(0,s)} R \xrightarrow{(s,0)} R \xrightarrow{i_2} R_2 \longrightarrow 0, \tag{3}$$

where the syzygies are the submodules $s\mathbb{Z} \times 0 \simeq R_1$ and $0 \times s\mathbb{Z} \simeq R_2$. It is easily seen that the short exact sequences

$$0 \longrightarrow 0 \times s\mathbb{Z} \hookrightarrow R \xrightarrow{i_1} R_1 \longrightarrow 0,$$

$$0 \longrightarrow s\mathbb{Z} \times 0 \hookrightarrow R \xrightarrow{i_2} R_2 \longrightarrow 0$$

do not split. Hence the R-modules R_k are not projective. By [2, Theorem 3.2.7], they are not flat either. It follows that $\operatorname{pd}_R R_k = \operatorname{fd}_R R_k = \infty$ and $\operatorname{lgld} R = \operatorname{wd} R = \infty$. At the same time, $\operatorname{lgld} R_k = \operatorname{wd} R_k = 1$. We see that it is impossible to estimate $\operatorname{lgld} R$ and $\operatorname{wd} R$ with only $\operatorname{lgld} R_k$ and $\operatorname{wd} R_k$ given.

Let M be an R-module of finite projective dimension with a projective resolution

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0. \qquad (4)$$

Since $\operatorname{lgld} R_k = 1$, we have $\operatorname{pd}_{R_k}(\operatorname{Tor}_0^R(R_k, M)) \leqslant 1$. Applying [2, Exercise 2.4.3] to the projective resolutions (2), (3) and (4), we obtain for a sufficiently large t

$$\operatorname{Tor}_1^R(R_k, M) \simeq \operatorname{Tor}_{1+2t}^R(R_k, M) \simeq \operatorname{Tor}_1^R(R_k, 0) = 0.$$

Consequently, $\operatorname{pd}_{R_k}(\operatorname{Tor}_1^R(R_k, M)) = \operatorname{pd}_{R_k} 0 = 0$. Theorem 2 now yields that $\operatorname{lFPD} R \leqslant 1$. In the same manner we can use Theorems 1 and 3 to show that $\operatorname{lFID} R \leqslant 1$ and $\operatorname{lFFD} R \leqslant 1$.

Consider the following projective resolution of the R-module $R/(s,s)R$:

$$0 \longrightarrow R \xrightarrow{(s,s)} R \xrightarrow{pr} R/(s,s)R \longrightarrow 0.$$

Since this short exact sequence does not split, we have $\operatorname{pd}_R(R/(s,s)R) = \operatorname{fd}_R(R/(s,s)R) = 1$. This clearly forces $\operatorname{lFPD} R = \operatorname{lFFD} R = 1$.

The subgroup R of the free Abelian group $\mathbb{Z} \times \mathbb{Z}$ is a free Abelian group also, therefore, applying the functor $\operatorname{Hom}_{\mathbb{Z}}(R, -)$ to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, we obtain a short exact sequence of R-modules

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

This sequence does not split and, by [2, Corollary 2.3.11], it is an injective resolution of the R-module $\operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Z})$. It follows that $\operatorname{id}_R(\operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Z})) = 1$, and hence that $\operatorname{lFID} R = 1$.

Example 2. Let F be a field. Define $R' = F(x,y)$, $R_1 = F(x)[y]$, $R_2 = F(y)[x]$, $R = R_1 \cap R_2 = F[x,y]$. Then the ring R is the pullback of R_1 and R_2 over R' in the commutative square (1) with inclusions i_k and j_k none of which is surjective. We claim that in this case our results are not true.

By [2, Proposition 4.1.5, Corollary 4.3.8], we have $\operatorname{wd} R = \operatorname{lgld} R = 2$ and $\operatorname{wd} R_k = \operatorname{lgld} R_k = 1$. Since these dimensions are finite, we have that

$\operatorname{lFFD} R = \operatorname{lFPD} R = \operatorname{lFID} R = 2$ and $\operatorname{lFFD} R_k = \operatorname{lFPD} R_k = \operatorname{lFID} R_k = 1$. It is easy to check that the R-modules R_k are flat. So the assumptions of Theorems 2 and 3 hold for $n = 1$, but their conclusions are false. The same observation can be made about the estimates [1, Proposition 5, 6 and 7, Theorems 9 and 10, Corollaries 12 and 13], which are not true in this case either.

It can be explained by the fact that the surjectivity condition cannot be dropped in the basic result [1, Theorem 1]. Indeed, we see at once that the R-module $M = R/(xR + yR)$ is neither projective nor flat, whilst the R_k-modules $R_k \bigotimes_R M = 0$ are projective and flat. From [2, Proposition 3.2.4] we conclude that the R-module $X = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is not injective, though the R_k-modules $\operatorname{Hom}_R(R_k, X) \simeq \operatorname{Hom}_{\mathbb{Z}}(R_k \bigotimes_R M, \mathbb{Q}/\mathbb{Z}) = 0$ are injective.

Acknowledgement. The author would like to thank Professor A. I. Generalov for suggesting the problem and guidance.

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