THE SECOND WEIGHTED MOMENT OF ζ

by

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Abstract. — We give an explicit formula for the second weighted moment of $\zeta(s)$ on the critical line tailored for fast computations with any desired accuracy.

Résumé. — Nous donnons une formule explicite pour le second moment pondéré de la fonction ζ sur la droite critique permettant une évaluation rapide pour n'importe quel degré de précision.

1. Introduction

For $\Im z > 0$ let

$$E_1(z) = 1 - 4\sum_{n=1}^{\infty} d(n)e(nz)$$

and let

$$E_1(z) - (1/z)E_1(-1/z) =: \psi(z).$$

Then ψ is analytic in $\mathbb{C}'$, the complex plane minus the negative real-axis and is given in that region by

$$\psi(z) = -2\frac{\log(2\pi z) - \gamma}{\pi i z} - \frac{2}{\pi}\int_{(-1/2)} \frac{\zeta(s)\zeta(1-s)}{\sin \pi s} z^{-s}\, ds.$$

Moreover, ψ satisfies a 3-term relation

$$\psi(z+1) = \psi(z) - \frac{1}{z+1}\psi\left(\frac{z}{z+1}\right)$$

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and for $|z| \leq 1$ satisfies

$$\frac{\pi i}{2}\psi(1+z) = \frac{-\log(1+z)-1}{1+z} + \frac{1}{1+z}\sum_{m=1}^{\infty} a_m(-1)^m z^m$$

where, for $m \geq 1$,

$$a_m = \frac{-1}{m+1} + 2b_m + 2\sum_{j=0}^{m-2}\binom{m-1}{j} b_{j+2}$$

with $b_1 = 0$ and

$$b_n = \frac{\zeta(n)B_n}{n}$$

for $n \geq 2$. The a_m are extremely small:

$$a_m = 2^{5/4}\pi^{3/4}\frac{e^{-2\sqrt{\pi m}}}{m^{3/4}}\left(\sin(2\sqrt{\pi m}+\frac{3}{8}\pi) + O\left(\frac{1}{\sqrt{m}}\right)\right)$$

so that the series $\sum a_m z^m$ converges everywhere on its circle of convergence $|z| = 1$. All of these results can be found in [**BC**].

2. The second moment of ζ

Theorem 1. — *Let*

$$I(\delta) = \int_0^\infty |\zeta(1/2+it)|^2 e^{-\delta t}\, dt.$$

Then, for $0 < \Re(\delta) < \pi$ and $\Im\delta \leq 0$ we have

$$I(\delta) = C_0(\delta) + C_1(\delta) + C_2(\delta) + C_3(\delta)$$

where

$$C_0(\delta) = -\frac{(\log(1-e^{-i\delta})+1)}{2\sin\frac{\delta}{2}} + \frac{\frac{\pi}{2}e^{-i\delta/2}}{1-e^{i\delta}} + e^{i\delta/2}(-\delta+\pi/2+i\gamma-i\log 2\pi);$$

$$C_1(\delta) = \frac{1}{2\sin\frac{\delta}{2}}\sum_{m=1}^{\infty} a_m e^{-im\delta},$$

where $a_m = \frac{-1}{m+1} + 2b_m + 2\sum_{j=0}^{m-2}\binom{m-1}{j} b_{j+2}$ with $b_1 = 0$ and $b_n = \frac{\zeta(n)B_n}{n}$ for $n \geq 2$;

$$C_2(\delta) = -\frac{i\pi}{\sin\delta/2}\sum_{n=1}^{\infty}(-1)^n d(n) e^{-\pi n\cot\frac{\delta}{2}};$$

and

$$C_3(\delta) = \int_0^\infty |\zeta(1/2+it)|^2 \frac{-e^{-\pi t}\sinh t\delta + i\cosh t\delta}{\cosh \pi t}\, dt.$$

If δ is real this simplifies to

$$I(\delta) = -\frac{1+\log(2\sin\frac{\delta}{2})}{2\sin\frac{\delta}{2}} + (\pi-\delta)\cos\frac{\delta}{2} + (\log 2\pi - \gamma)\sin\frac{\delta}{2}$$
$$+\frac{1}{2\sin\frac{\delta}{2}}\sum_{m=1}^{\infty} a_m \cos m\delta - \int_0^{\infty} |\zeta(1/2+it)|^2 \frac{e^{-\pi t}\sinh t\delta}{\cosh \pi t}\, dt.$$

We can use the fact that

$$\sum_{m=1}^{\infty} a_m = \gamma + 1 - \log 2\pi$$

and that

$$\frac{\cos m\delta - 1}{2\sin\frac{\delta}{2}} = -U_{m-1}(\cos\frac{\delta}{2})\sin\frac{m\delta}{2},$$

where U_m is the mth Chebyshev polynomial, to rewrite this further.

Corollary 1. — *Suppose that $0 < \delta < 2\pi$. Then*

$$I(\delta) = \frac{\gamma - \log 2\pi - \log(2\sin\frac{\delta}{2})}{2\sin\frac{\delta}{2}} + (\pi-\delta)\cos\frac{\delta}{2} + (\log 2\pi - \gamma)\sin\frac{\delta}{2}$$
$$-\sum_{m=1}^{\infty} a_m U_{m-1}(\cos\frac{\delta}{2})\sin\frac{m\delta}{2} - \int_0^{\infty} |\zeta(1/2+it)|^2 \frac{e^{-\pi t}\sinh t\delta}{\cosh \pi t}\, dt.$$

In this version the first term above is the readily recognized usual main term. Note that the integral in the right-hand side of this formula can be rewritten in terms of the original integral I:

$$-\int_0^{\infty} |\zeta(1/2+it)|^2 \frac{e^{-\pi t}\sinh t\delta}{\cosh \pi t}\, dt = \sum_{n=1}^{\infty}(-1)^{n+1}\left(I(2\pi n+\delta) - I(2\pi n - \delta)\right).$$

Another way to put it is if we let

$$w(\delta,t) := e^{-\delta t} + \frac{e^{-\pi t}\sinh \delta t}{\cosh \pi t} = \frac{\cosh(\pi-\delta)t}{\cosh \pi t},$$

then

$$\int_0^{\infty} |\zeta(1/2+it)|^2 w(\delta,t)\, dt = \frac{\gamma - \log 2\pi - \log(2\sin\frac{\delta}{2})}{2\sin\frac{\delta}{2}} + (\frac{\pi}{2}-\delta)\cos\frac{\delta}{2} + (\log 2\pi - \gamma)\sin\frac{\delta}{2}$$
$$+\sum_{m=1}^{\infty} a_m U_{m-1}(\cos\frac{\delta}{2})\sin\frac{m\delta}{2}.$$

Since $a_m \ll e^{-\sqrt{m}}$ we have the following estimate for a precise evaluation of $I(\delta)$ as $\delta \to 0$.

Corollary 2. — *Given $\delta > 0$ and $N \geq 1$ we can compute $I(\delta)$ to an accuracy of 10^{-N} in time $t(\delta, N) \ll N^2$.*

This is fairly remarkable in that it doesn't depend on δ! By contrast if one considers the associated integral

$$\int_0^{\frac{1}{\delta}} |\zeta(1/2+it)|^2 \, dt$$

then the time to calculate will depend on δ in a significant way, say $\delta^{-\theta}$ for some $\theta > 0$. Finally, using the fact that when δ is real, the imaginary part of $C_3(\delta)$ involves the weight $w(\pi-\delta,t)$ we deduce a formula for the divisor sum in $C_2(\delta)$ in terms of the coefficients a_m:

Corollary 3. — *For* $0<\delta<\pi$

$$\frac{\pi}{\sin\frac{\delta}{2}}\sum_{n=1}^{\infty} d(n)(-1)^n e^{-n\pi\cot\frac{\delta}{2}} = -\frac{1}{\sin\delta}\sum_{m=1}^{\infty}(a_{2m}+a_{2m-1})\sin((2m-\frac{1}{2})\delta) + \frac{\delta}{4\sin\frac{\delta}{2}} - \frac{1+\log(2\cos\frac{\delta}{2})}{2\cos\frac{\delta}{2}}.$$

This can be rewritten as

$$\sum_{m=1}^{\infty}(a_{2m}+a_{2m-1})\sin((2m-\frac{1}{2})\delta) = \frac{\delta\cos\frac{\delta}{2}}{2} - \sin\frac{\delta}{2}\left(1+\log(2\cos\frac{\delta}{2})\right) - 2\pi\cos\frac{\delta}{2}\sum_{n=1}^{\infty} d(n)(-1)^n e^{-n\pi\cot\frac{\delta}{2}}.$$

If $\delta = \frac{1}{1000}$, then the first term of the divisor sum on the right-hand side of the formula is $-1.12\cdots\times 10^{-2729}$, whereas the first two terms on the right-hand side are $-0.000159155\ldots$. That means that the sum on the left-hand side is $-0.000159155\cdots - 1.12\cdots\times 10^{-2729}$ up to an error of around $-1.99\cdots\times 10^{-5458}$, which is the second term of the divisor sum. Since the terms a_{2m} are around $e^{-2\sqrt{2\pi m}}$ it would take more than 6 million terms of the series in m (each with thousands of digits of accuracy) to numerically check this.

3. Proof of Theorem 1

Assume first of all that δ is real with $0<\delta<\pi$. We have

$$I(\delta) = \frac{1}{i}\int_{1/2}^{1/2+i\infty}\zeta(s)\zeta(1-s)e^{i\delta(s-1/2)}\,ds = \frac{e^{-i\delta/2}}{i}\int_{1/2}^{1/2+i\infty}\chi(1-s)\zeta(s)^2 e^{i\delta s}\,ds.$$

Now

$$\chi(1-s) = (2\pi)^{-s}\Gamma(s)(e^{\pi i s/2}+e^{-\pi i s/2})$$

so that

$$I(\delta) = I_1(\delta) + I_2(\delta)$$

where

$$I_1(\delta) = \frac{e^{-i\delta/2}}{i}\int_{1/2}^{1/2+i\infty}(2\pi)^{-s}\Gamma(s)e^{-\pi i s/2}\zeta(s)^2 e^{i\delta s}\,ds$$

and

$$I_2(\delta) = \frac{e^{-i\delta/2}}{i} \int_{1/2}^{1/2+i\infty} (2\pi)^{-s}\Gamma(s)e^{\pi is/2}\zeta(s)^2 e^{i\delta s}\ ds.$$

Now

$$I_1(\delta) = I_0(\delta) - I_3(\delta)$$

where

$$I_0(\delta) = \frac{e^{-i\delta/2}}{i} \int_{1/2-i\infty}^{1/2+i\infty} (2\pi)^{-s}\Gamma(s)e^{-\pi is/2}\zeta(s)^2 e^{i\delta s}\ ds$$

and

$$I_3(\delta) = \frac{e^{-i\delta/2}}{i} \int_{1/2-i\infty}^{1/2} (2\pi)^{-s}\Gamma(s)e^{-\pi is/2}\zeta(s)^2 e^{i\delta s}\ ds.$$

Thus, $I(\delta) = I_0(\delta) + I_2(\delta) - I_3(\delta)$. Note that $I_2(\delta)$ and $I_3(\delta)$ are analytic for $|\delta| < \pi/2$. We rewrite I_2 and I_3 as integrals over t as

$$\begin{aligned} I_2(\delta) &= \frac{e^{-i\delta/2}}{i} \int_{1/2}^{1/2+i\infty} \zeta(s)\zeta(1-s)\frac{e^{\pi is/2}}{2\cos \pi s/2}e^{i\delta s}\ ds \\ &= \int_0^\infty |\zeta(1/2+it)|^2 e^{-\pi t/2}\frac{e^{\pi i/4}e^{-\delta t}}{2\cos\frac{\pi}{2}(1/2+it)}\ dt \end{aligned}$$

and

$$\begin{aligned} I_3(\delta) &= \frac{e^{-i\delta/2}}{i} \int_{1/2-i\infty}^{1/2} \zeta(s)\zeta(1-s)\frac{e^{-\pi is/2}}{2\cos \pi s/2}e^{i\delta s}\ ds \\ &= \frac{e^{-i\delta/2}}{i} \int_{1/2}^{1/2+i\infty} \zeta(s)\zeta(1-s)\frac{e^{-\pi i(1-s)/2}}{2\cos \pi(1-s)/2}e^{i\delta(1-s)}\ ds \\ &= \int_0^\infty |\zeta(1/2+it)|^2 e^{-\pi t/2}\frac{e^{-\pi i/4}e^{\delta t}}{2\cos\frac{\pi}{2}(1/2-it)}\ dt. \end{aligned}$$

Next we write $e^{\delta t} = \cosh \delta t + \sinh \delta t$ and $e^{-\delta t} = \cosh \delta t - \sinh \delta t$. Also, for real t,

$$\frac{e^{\pi i/4}}{2\cos\frac{\pi}{2}(1/2+it)} + \frac{e^{-\pi i/4}}{2\cos\frac{\pi}{2}(1/2-it)} = \frac{e^{-\pi t/2}}{\cosh \pi t}$$

and

$$\frac{e^{\pi i/4}}{2\cos\frac{\pi}{2}(1/2+it)} - \frac{e^{-\pi i/4}}{2\cos\frac{\pi}{2}(1/2-it)} = i\frac{e^{\pi t/2}}{\cosh \pi t}.$$

Thus,

$$I_2(\delta) - I_3(\delta) = \int_0^\infty |\zeta(1/2+it)|^2 \frac{-e^{-\pi t}\sinh t\delta + i\cosh t\delta}{\cosh \pi t}\ dt.$$

Returning to I_0 we have

$$\begin{aligned} I_0(\delta) &= -2\pi e^{-i\delta/2}\operatorname*{Res}_{s=1}(2\pi)^{-s}\Gamma(s)e^{-\pi is/2}\zeta(s)^2 e^{i\delta s} + J(\delta) \\ &= e^{i\delta/2}(-\delta + \pi/2 + i\gamma - i\log 2\pi) + J(\delta) \end{aligned}$$

where

$$J(\delta) = \frac{e^{-i\delta/2}}{i}\int_{2-i\infty}^{2+i\infty}(2\pi)^{-s}\Gamma(s)e^{-\pi is/2}\zeta(s)^2e^{i\delta s}\ ds;$$

note that this is a place where we need the (temporary) assumption that δ is real to ensure convergence of the integral on the new path. Expanding $\zeta(s)^2 = \sum_{n=1}^{\infty} d(n)n^{-s}$ into its Dirichlet series and interchanging the summation and integration, we obtain

$$\begin{aligned} J(\delta) &= 2\pi e^{-i\delta/2}\sum_{n=1} d(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)(2\pi ine^{-i\delta})^{-s}\ ds \\ &= 2\pi e^{-i\delta/2}\sum_{n=1}^{\infty} d(n)e^{-2\pi ine^{-i\delta}} \\ &= \frac{\pi}{2}e^{-i\delta/2}(1-E_1(-e^{-i\delta})). \end{aligned}$$

Now

$$E_1(-e^{-i\delta}) = E_1(1-e^{-i\delta}) = \frac{1}{1-e^{-i\delta}}E_1\left(\frac{-1}{1-e^{-i\delta}}\right) + \psi(1-e^{-i\delta})$$

and this is

$$\frac{1}{1-e^{-i\delta}}\left(1-4\sum_{n=1}^{\infty} d(n)e\left(\frac{-n}{1-e^{-i\delta}}\right)\right) + \psi(1-e^{-i\delta}).$$

Thus,

$$\begin{aligned} J(\delta) &= \frac{\pi}{2}e^{-i\delta/2} - \frac{\frac{\pi}{2}e^{-i\delta/2}}{1-e^{-i\delta}}\left(1-4\sum_{n=1}^{\infty} d(n)e\left(\frac{-n}{1-e^{-i\delta}}\right)\right) - \frac{\pi}{2}e^{-i\delta/2}\psi(1-e^{-i\delta}) \\ &= \frac{\frac{\pi}{2}e^{-i\delta/2}}{1-e^{i\delta}} - \frac{i\pi}{\sin\delta/2}\sum_{n=1}^{\infty}(-1)^n d(n)e^{-\pi n\cot\frac{\delta}{2}} - \frac{\pi}{2}e^{-i\delta/2}\psi(1-e^{-i\delta}). \end{aligned}$$

Altogether we now have

$$\begin{aligned} \int_0^{\infty}|\zeta(1/2+it)|^2e^{-\delta t}\ dt &= \frac{\frac{\pi}{2}e^{-i\delta/2}}{1-e^{i\delta}} + e^{i\delta/2}(-\delta+\pi/2+i\gamma-i\log 2\pi) \\ &\quad -\frac{\pi}{2}e^{-i\delta/2}\psi(1-e^{-i\delta}) \\ &\quad -\frac{i\pi}{\sin\delta/2}\sum_{n=1}^{\infty}(-1)^n d(n)e^{-\pi n\cot\frac{\delta}{2}} \\ &\quad +\int_0^{\infty}|\zeta(1/2+it)|^2\frac{-e^{-\pi t}\sinh t\delta + i\cosh t\delta}{\cosh\pi t}\ dt. \end{aligned}$$

Recall that

$$\psi(z) = -\frac{2(\log z+1)}{\pi iz} + \frac{2}{\pi iz}\sum_{m=1}^{\infty} a_m(-1)^m(z-1)^m$$

so that

$$\psi(1-e^{-i\delta}) = -\frac{2(\log(1-e^{-i\delta})+1)}{\pi i(1-e^{-i\delta})} + \frac{2}{\pi i(1-e^{-i\delta})}\sum_{m=1}^{\infty} a_m e^{-im\delta}$$

and

$$-\frac{\pi}{2}e^{-i\delta/2}\psi(1-e^{-i\delta}) = -\frac{(\log(1-e^{-i\delta})+1)}{2\sin\frac{\delta}{2}} + \frac{1}{2\sin\frac{\delta}{2}}\sum_{m=1}^{\infty} a_m e^{-im\delta}.$$

The assertion of the theorem now follows for real δ. But both sides are analytic in the region $0 < \Re < \pi$ and $\Im\delta < 0$. Therefore, by analytic continuation the identity of the theorem holds in this larger region of the complex plane.

References

[BC] Bettin, Sandro; Conrey, Brian, Period functions and cotangent sums. Algebra Number Theory **7** (2013), no. 1, 215–242.

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